Detailed Derivations of Vandermonde Equation of Finite Difference

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Equation:

$$\begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1^2 & 2^2 & 3^2 & ... & N^2 \\ 1^4 & 2^4 & 3^4 & ... & N^4 \\ 1^6 & 2^6 & 3^6 & ... & N^6 \\ ... \\ 1^{2N-2} & 2^{2N-2} & 3^{2N-2} & ... & N^{2N-2} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ ... \\ C_N \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 1^0 & 2^0 & 3^0 & 4^0 & 5^0 & 6^0 & ... & N^0 & 1 \\ 1^2 & 2^2 & 3^2 & 4^2 & 5^2 & 6^2 & ... & N^2 & 0 \\ 1^4 & 2^4 & 3^4 & 4^4 & 5^4 & 6^4 & ... & N^4 & 0 \\ 1^6 & 2^6 & 3^6 & 4^6 & 5^6 & 6^6 & ... & N^6 & 0 \\ 1^8 & 2^8 & 3^8 & 4^8 & 5^8 & 6^8 & ... & N^8 & 0 \\ ... \\ 1^{2N-2} & 2^{2N-2} & 3^{2N-2} & ... & ... & ... & ... & N^{2N-2} & 0 \end{bmatrix} \tag{2}$$

(1) Step 1: $row(i+1) - row(i)[i = 1, 2, 3, 4, 5, ..., N-1]$

$$\begin{bmatrix} 1^0 & 2^0 & 3^0 & 4^0 & 5^0 & 6^0 & ... & N^0 & 1 \\ 0 & 2^2-1 & 3^2-1 & 4^2-1 & 5^2-1 & 6^2-1 & ... & N^2-1 & -1 \\ 0 & 2^2(2^2-1) & 3^2(3^2-1) & 4^2(4^2-1) & 5^2(5^2-1) & 6^2(6^2-1) & ... & N^2(N^2-1) & 0 \\ 0 & 2^4(2^2-1) & 3^4(3^2-1) & 4^4(4^2-1) & 5^4(5^2-1) & 6^4(6^2-1) & ... & N^4(N^2-1) & 0 \\ 0 & 2^6(2^2-1) & 3^6(3^2-1) & 4^6(4^2-1) & 5^6(5^2-1) & 6^6(6^2-1) & ... & N^6(N^2-1) & 0 \\ ... \\ 0 & 2^{2N-4}(2^2-1) & 3^{2N-4}(3^2-1) & 4^{2N-4}(4^2-1) & 5^{2N-4}(5^2-1) & 6^{2N-4}(6^2-1) & ... & N^{2N-4}(N^2-1) & 0 \end{bmatrix} \tag{3}$$

(2) Step2: $row(i+1) - 2^2 row(i)[i = 2, 3, 4, 5, ..., N-1]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2^2-1 & 3^2-1 & 4^2-1 & 5^2-1 & 6^2-1 & N^2-1 & -1 \\ 0 & 0 & (3^2-2^2)(3^2-1) & (4^2-2^2)(4^2-1) & (5^2-2^2)(5^2-1) & (6^2-2^2)(6^2-1) & (N^2-2^2)(N^2-1) & 2^2 \\ 0 & 0 & 3^2(3^2-2^2)(3^2-1) & 4^2(4^2-2^2)(4^2-1) & 5^2(5^2-2^2)(5^2-1) & 6^2(6^2-2^2)(6^2-1) & N^2(N^2-2^2)(N^2-1) & 0 \\ 0 & 0 & 3^4(3^2-2^2)(3^2-1) & 4^4(4^2-2^2)(4^2-1) & 5^4(5^2-2^2)(5^2-1) & 6^4(6^2-2^2)(6^2-1) & N^4(N^2-2^2)(N^2-1) & 0 \\ ... \\ 0 & 0 & 3^{2N-6}(3^2-2^2)(3^2-1) & 4^{2N-6}(4^2-2^2)(4^2-1) & 5^{2N-6}(5^2-2^2)(5^2-1) & 6^{2N-6}(6^2-2^2)(6^2-1) & N^{2N-6}(N^2-2^2)(N^2-1) & 0 \end{bmatrix} \tag{4}$$

(3) Step3: $row(i+1) - 3^2 row(i)[i = 3, 4, 5, ..., N-1]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2^2-1 & 3^2-1 & 4^2-1 & 5^2-1 & 6^2-1 & N^2-1 & -1 \\ 0 & 0 & (3^2-2^2)(3^2-1) & (4^2-2^2)(4^2-1) & (5^2-2^2)(5^2-1) & (6^2-2^2)(6^2-1) & (N^2-2^2)(N^2-1) & 2^2 \\ 0 & 0 & 0 & (4^2-3^2)(4^2-2^2)(4^2-1) & (5^2-3^2)(5^2-2^2)(5^2-1) & (6^2-3^2)(6^2-2^2)(6^2-1) & (N^2-3^2)(N^2-2^2)(N^2-1) & -2^2 3^2 \\ 0 & 0 & 0 & 4^2(4^2-3^2)(4^2-2^2)(4^2-1) & 5^2(5^2-3^2)(5^2-2^2)(5^2-1) & 6^2(6^2-3^2)(6^2-2^2)(6^2-1) & N^2(N^2-3^2)(N^2-2^2)(N^2-1) & 0 \\ ... \\ 0 & 0 & 0 & 4^{2N-8}(4^2-3^2)(4^2-2^2)(4^2-1) & 5^{2N-8}(5^2-3^2)(5^2-2^2)(5^2-1) & 6^{2N-8}(6^2-3^2)(6^2-2^2)(6^2-1) & N^{2N-8}(N^2-3^2)(N^2-2^2)(N^2-1) & 0 \end{bmatrix} \tag{5}$$

(3) Step4: $row(i+1) - 4^2 row(i)[i = 4, 5, ..., N-1]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2^2-1 & 3^2-1 & 4^2-1 & 5^2-1 & 6^2-1 & N^2-1 & -1 \\ 0 & 0 & (3^2-2^2)(3^2-1) & (4^2-2^2)(4^2-1) & (5^2-2^2)(5^2-1) & (6^2-2^2)(6^2-1) & (N^2-2^2)(N^2-1) & 2^2 \\ 0 & 0 & 0 & (4^2-3^2)(4^2-2^2)(4^2-1) & (5^2-3^2)(5^2-2^2)(5^2-1) & (6^2-3^2)(6^2-2^2)(6^2-1) & (N^2-3^2)(N^2-2^2)(N^2-1) & -2^2 3^2 \\ 0 & 0 & 0 & 0 & (5^2-4^2)(5^2-3^2)(5^2-2^2)(5^2-1) & (6^2-4^2)(6^2-3^2)(6^2-2^2)(6^2-1) & (N^2-4^)(N^2-3^2)(N^2-2^2)(N^2-1) & 2^2 3^2 4^2 \\ 0 & 0 & 0 & 0 & 5^2(5^2-4^2)(5^2-3^2)(5^2-2^2)(5^2-1) & 6^2(6^2-4^2)(6^2-3^2)(6^2-2^2)(6^2-1) & N^2(N^2-4^)(N^2-3^2)(N^2-2^2)(N^2-1) & 0 \\ ... \\ 0 & 0 & 0 & 0 & 5^{2N-10}5^2-4^2(5^2-3^2)(5^2-2^2)(5^2-1) & 6^{2N-10}(6^2-4^2)(6^2-3^2)(6^2-2^2)(6^2-1) & N^{2N-10}(N^2-4^2)(N^2-3^2)(N^2-2^2)(N^2-1) & 0 \end{bmatrix} \tag{6}$$

(k) Step k: $row(i+1) - k^2 row(i)[i = k, k+1, ..., N-1]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & ... & 1 & 1 \\ 0 & 2^2-1 & 3^2-1 & 4^2-1 & 5^2-1 & 6^2-1 & ... & N^2-1 & -1 \\ 0 & 0 & (3^2-2^2)(3^2-1) & (4^2-2^2)(4^2-1) & (5^2-2^2)(5^2-1) & (6^2-2^2)(6^2-1) & ... & (N^2-2^2)(N^2-1) & 2^2 \\ 0 & 0 & 0 & (4^2-3^2)(4^2-2^2)(4^2-1) & (5^2-3^2)(5^2-2^2)(5^2-1) & (6^2-3^2)(6^2-2^2)(6^2-1) & ... & (N^2-3^2)(N^2-2^2)(N^2-1) & -2^2 3^2 \\ 0 & 0 & 0 & 0 & (5^2-4^2)(5^2-3^2)(5^2-2^2)(5^2-1) & (6^2-4^2)(6^2-3^2)(6^2-2^2)(6^2-1) & ... & (N^2-4^)(N^2-3^2)(N^2-2^2)(N^2-1) & 2^2 3^2 4^2 \\ ... \\ 0 & 0 & 0 & 0 & ...0 & ((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1) & ((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1) & (N^2-k^2)(N^2-(k-1)^2)...(N^2-1) & (-1)^k 2^2 3^2 4^2 ... k^2 \\ 0 & 0 & 0 & 0 & ...0 & ((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1) & ((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1) & (N^2-k^2)(N^2-(k-1)^2)...(N^2-1) & (-1)^k 2^2 3^2 4^2 ... k^2 \\ 0 & 0 & 0 & 0 & ...0 & (k+1)^2((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1) & (k+2)^2((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1) & N^2(N^2-k^2)(N^2-(k-1)^2)...(N^2-1) & 0 \\ 0 & 0 & 0 & 0 & ...0 & (k+1)^4((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1) & (k+2)^4((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1) & N^4(N^2-k^2)(N^2-(k-1)^2)...(N^2-1) & 0 \\ 0 & 0 & 0 & 0 & ...0 & (k+1)^{2N-2k-2}((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1) & (k+2)^{2N-2k-2}((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1) & N^{2N-2k-2}(N^2-k^2)(N^2-(k-1)^2)...(N^2-1) & 0 \end{bmatrix} \tag{7}$$

(N-1) Step N-1: $row(i+1) - k^2 row(i)[i = N-1]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & ... & 1 & 1 \\ 0 & 2^2-1 & 3^2-1 & 4^2-1 & 5^2-1 & 6^2-1 & ... & N^2-1 & -1 \\ 0 & 0 & (3^2-2^2)(3^2-1) & (4^2-2^2)(4^2-1) & (5^2-2^2)(5^2-1) & (6^2-2^2)(6^2-1) & ... & (N^2-2^2)(N^2-1) & 2^2 \\ 0 & 0 & 0 & (4^2-3^2)(4^2-2^2)(4^2-1) & (5^2-3^2)(5^2-2^2)(5^2-1) & (6^2-3^2)(6^2-2^2)(6^2-1) & ... & (N^2-3^2)(N^2-2^2)(N^2-1) & -2^2 3^2 \\ 0 & 0 & 0 & 0 & (5^2-4^2)(5^2-3^2)(5^2-2^2)(5^2-1) & (6^2-4^2)(6^2-3^2)(6^2-2^2)(6^2-1) & ... & (N^2-4^)(N^2-3^2)(N^2-2^2)(N^2-1) & 2^2 3^2 4^2 \\ ... \\ 0 & 0 & 0 & ...0 & ((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1) & ((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1) & ... & (N^2-k^2)(N^2-(k-1)^2)...(N^2-1) & (-1)^k 2^2 3^2 4^2 ... k^2 \\ 0 & 0 & 0 & ...0 & 0 & ((k+2)^2-(k+1)^2)((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1) & ... & (N^2-(k+1)^2)(N^2-k^2)(N^2-(k-1)^2)...(N^2-1) & (-1)^{k+1} 2^2 3^2 4^2 ... (k+1)^2 \\ ... \\ 0 & 0 & 0 & ...0 & 0 & ...0 & ((N-1)^2-(N-2)^2)...(N^2-3^2)(N^2-2^2)(N^2-1) & (N^2-(N-2)^2)...(N^2-3^2)(N^2-2^2)(N^2-1) & (-1)^{N-2} 2^2 3^2 4^2 ... (N-2)^2 \\ 0 & 0 & 0 & ...0 & 0 & ...0 & 0 & (N^2-(N-1)^2)...(N^2-3^2)(N^2-2^2)(N^2-1) & (-1)^{N-1} 2^2 3^2 4^2 ... (N-1)^2 \end{bmatrix} \tag{8}$$

$$a_{ij} = (j^2-(i-1)^2)(j^2-(i-2)^2)(j^2-1) \tag{9}$$

(N) Step N-1: Divide: $row(i+1)/a_{ii}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & ... & 1 & 1 \\ 0 & 1 & \frac{3^2-1}{2^2-1} & \frac{4^2-1}{2^2-1} & \frac{5^2-1}{2^2-1} & \frac{6^2-1}{2^2-1} & ... & \frac{N^2-1}{2^2-1} & \frac{-1}{2^2-1} \\ 0 & 0 & 1 & \frac{(4^2-2^2)(4^2-1)}{(3^2-2^2)(3^2-1)} & \frac{(5^2-2^2)(5^2-1)}{(3^2-2^2)(3^2-1)} & \frac{(6^2-2^2)(6^2-1)}{(3^2-2^2)(3^2-1)} & ... & \frac{(N^2-2^2)(N^2-1)}{(3^2-2^2)(3^2-1)} & \frac{2^2}{(3^2-2^2)(3^2-1)} \\ ... \\ 0 & 0 & 0 & ...0 & 1 & \frac{((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1)}{((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1)} & ... & \frac{(N^2-k^2)(N^2-(k-1)^2)...(N^2-1)}{((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1)} & \frac{(-1)^k 2^2 3^2 4^2 ... k^2}{((k+1)^2-k^2)((k+1)^2-(k-1)^2)...((k+1)^2-1)} \\ 0 & 0 & 0 & ...0 & 0 & 1 & ... & \frac{(N^2-(k+1)^2)(N^2-k^2)(N^2-(k-1)^2)...(N^2-1)}{((k+2)^2-(k+1)^2)((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1)} & \frac{(-1)^{k+1} 2^2 3^2 4^2 ... (k+1)^2}{((k+2)^2-(k+1)^2)((k+2)^2-k^2)((k+2)^2-(k-1)^2)...((k+2)^2-1)} \\ ... \\ 0 & 0 & 0 & ...0 & 0 & ...0 & 1 & \frac{(N^2-(N-2)^2)...(N^2-3^2)(N^2-2^2)(N^2-1)}{((N-1)^2-(N-2)^2)...(N^2-3^2)(N^2-2^2)(N^2-1)} & \frac{(-1)^{N-2} 2^2 3^2 4^2 ... (N-2)^2}{((N-1)^2-(N-2)^2)...(N^2-3^2)(N^2-2^2)(N^2-1)} \\ 0 & 0 & 0 & ...0 & 0 & ...0 & 0 & 1 & \frac{(-1)^{N-1} 2^2 3^2 4^2 ... (N-1)^2}{(N^2-(N-1)^2)...(N^2-3^2)(N^2-2^2)(N^2-1)} \end{bmatrix} \tag{10}$$

$$\begin{aligned} a_{ij} &= \frac{[(j^2-(i-1)^2)][j^2-(i-2)^2]...[j^2-1]}{[(i^2-(i-1)^2)][i^2-(i-2)^2]...[i^2-1]}(j \geq i) && (11) \\ &= \frac{(j+i-1)(j-i+1)(j+i-2)(j-i+2)(j+i-3)(j-i+3)...(j+1)(j-1)}{(i+i-1)(i-i+1)(i+i-2)(i-i+2)(i+i-3)(i-i+3)...(i+1)(i-1)} && (12) \\ &= \frac{(j+i-1)(j+i-2)...(j+1)(j-1)(j-2)...(j-i+1)}{(2i-1)(2i-2)...(i+1)(i-1)(i-2)...1} && (13) \\ &= \frac{i(j+i-1)(j+i-2)...(j-i+1)}{j(2i-1)(2i-2)...1} && (14) \\ &= \frac{i(j+i-1)!}{j(2i-1)!(j-i)!} && (15) \\ b_i &= \frac{(-1)^{i-1} 2^2 3^2 ... (i-1)^2}{[(i^2-(i-1)^2)][i^2-(i-2)^2]...[i^2-1]} && (16) \\ &= \frac{(-1)^{i-1}[(i-1)!]^2}{(i+i-1)(i-i+1)(i+i-2)(i-i+2)...(i+1)(i-1)} && (17) \\ &= \frac{(-1)^{i-1}[(i-1)!]^2}{(i+i-1)(i+i-2)...(i+1)(i-1)...1} && (18) \\ &= \frac{(-1)^{i-1} i[(i-1)!]^2}{(2i-1)!} && (19) \\ &= \frac{(-1)^{i-1}(i-1)!i!}{(2i-1)!} && (20) \\ & && (21) \end{aligned}$$

$$\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} & ... & ... & a_{1N} \\ 0 & 1 & a_{23} & a_{24} & a_{25} & ... & ... & a_{2N} \\ 0 & 0 & 1 & a_{34} & a_{35} & ... & ... & a_{3N} \\ ... \\ 0 & 0 & 0 & ... & 1 & a_{k(k+1)} & ... & a_{kN} \\ ... \\ 0 & 0 & 0 & ... & 0 & 0 & ... & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ ... \\ C_k \\ ... \\ C_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ ... \\ b_k \\ ... \\ b_N \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ ... \\ C_k \\ ... \\ C_N \end{bmatrix} = -\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & ... & ... & a_{1N} \\ 0 & 0 & a_{23} & a_{24} & a_{25} & ... & ... & a_{2N} \\ 0 & 0 & 0 & a_{34} & a_{35} & ... & ... & a_{3N} \\ ... \\ 0 & 0 & 0 & ... & 0 & a_{k(k+1)} & ... & a_{kN} \\ ... \\ 0 & 0 & 0 & ... & 0 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ ... \\ C_k \\ ... \\ C_N \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ ... \\ b_k \\ ... \\ b_N \end{bmatrix} \tag{23}$$

$$\begin{aligned} C_k &= b_k - a_{k(k+1)}C_{k+1} - a_{k(k+2)}C_{k+2} - ... - a_{kN}C_N && (24) \\ &= b_k - \sum_{i=k+1}^{N} a_{ki}C_i && (25) \\ &= \frac{(-1)^{k-1}(k-1)!k!}{(2k-1)!} - \sum_{i=k+1}^{N} \frac{k(i+k-1)!}{i(2k-1)!(i-k)!}C_i && (26) \\ C_N &= \frac{(-1)^{N-1}(N-1)!N!}{(2N-1)!} && (27) \\ & && (28) \end{aligned}$$

$$\begin{aligned} C_N &= \frac{(-1)^{N-1}(N-1)!N!}{(2N-1)!} && (29) \\ C_2^2 &= \frac{(-1)^1 1!2!}{3!} = -\frac{1}{3} && (30) \\ C_3^3 &= \frac{(-1)^2 2!3!}{5!} = \frac{1}{10} && (31) \\ C_{N-1} &= \frac{(-1)^{N-2}(N-2)!(N-1)!}{(2N-3)!} - \frac{(N-1)(2N-2)!}{N(2N-3)!}C_N && (32) \\ & && (33) \end{aligned}$$